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LETTER TO THE EDITOR

Generalised operator reduction formula for multiple hypergeometric series ${}^N F(x_1, \ldots, x_N)$

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Abstract. The reduction formula obtained in a previous paper has been generalised as follows. If ${}^N F'$ and ${}^N F''$ are generalised hypergeometric series of N variables (GHS-N), then, the operation on ${}^N F''$ $(x_1 t_1, \ldots, x_n t_n)$, ${}^N F'(\partial/\partial t_1, \ldots, \partial/\partial t_N)$ gives, at $t_1 = t_2 = \ldots = t_N = 0$, a function ${}^N F(x_1, \ldots, x_N)$, which is, again, a GHS-N. The differentiation procedure can be regarded as an algebraic Ω-multiplication which gives rise to a group-theoretical interpretation of the method. A concept of Ω-equivalent relations has been introduced which allows systematisation of numerous results obtained in special functions theory. As the functions ${}^N F$ comprise a number of physically interesting series, the operator factorisation method seems to be applicable to many physical problems providing a possibility of reducing any ${}^N F$ to simpler functions of the same class.

As was shown in (Niukkanen 1983), the generalised hypergeometric series ${}^N F(x_1, \ldots, x_N)$ of N variables (GHS-N), defined by†

$$ {}^N F^{p_0, p_1 \ldots p_N}_{q_0, q_1 \ldots q_N} \left[\begin{matrix} \boldsymbol{a}_0; \boldsymbol{a}_1; \ldots; \boldsymbol{a}_N; x_1, \ldots, x_N \\ \boldsymbol{c}_0; \boldsymbol{c}_1; \ldots; \boldsymbol{c}_N \end{matrix} \right] = \sum_{i_1, \ldots, i_N} \frac{(\boldsymbol{a}_0)_{i_1 + \ldots i_N}}{(\boldsymbol{c}_0)_{i_1 + \ldots i_N}} \prod_{s=1}^{N} \frac{(\boldsymbol{a}_s)_{i_s}}{(\boldsymbol{c}_s)_{i_s}} \frac{x_s^{i_s}}{i_s!}, \tag{1} $$

allows remarkably simple operator representation

$$ {}^N F^{p_0, p_s}_{q_0, q_s} = F^{p_0}_{q_0} \left[\begin{matrix} \boldsymbol{a}_0; \partial/\partial t \\ \boldsymbol{c}_0 \end{matrix} \right] F^{p_1}_{q_1} \left[\begin{matrix} \boldsymbol{a}_1; x_1 t \\ \boldsymbol{c}_1 \end{matrix} \right] \ldots F^{p_N}_{q_N} \left[\begin{matrix} \boldsymbol{a}_N; x_N t \\ \boldsymbol{c}_N \end{matrix} \right] \Bigg|_{t=0}, \tag{2} $$

which relates the function ${}^N F$ to the product of *standard* GHS-1 F^p_q. Moreover the GHS-1 allow further reduction (let $N = 1$ in (2)):

$$ F^{p_0 + p_1}_{q_0 + q_1} \left[\begin{matrix} \boldsymbol{a}_0, \boldsymbol{a}_1; x \\ \boldsymbol{c}_0, \boldsymbol{c}_1 \end{matrix} \right] = F^{p_0}_{q_0} \left[\begin{matrix} \boldsymbol{a}_0; \partial/\partial t \\ \boldsymbol{c}_0 \end{matrix} \right] F^{p_1}_{q_1} \left[\begin{matrix} \boldsymbol{a}_1; xt \\ \boldsymbol{c}_1 \end{matrix} \right] \Bigg|_{t=0} \tag{3} $$

to a simpler GHS-1. This gives us an evident possibility of deducing the properties of complicated functions as corollaries of relations for simpler series.

† *Notation.* In accordance with Niukkanen (1983), the quantities p_σ and q_σ denote the numbers of components in numerator ($\boldsymbol{a}_\sigma$) and denominator ($\boldsymbol{c}_\sigma$) sets of parameters, respectively ($\boldsymbol{a}_0$ and $\boldsymbol{c}_0$ are the gluing parameters and $\boldsymbol{a}_s$ and $\boldsymbol{c}_s$ $(s = 1, 2, \ldots, N)$ are the 'individual' parameters). For brevity, $(\boldsymbol{a})_i = (a^1)_i \ldots (a^p)_i$ with $(a)_i = \Gamma(a+i)/\Gamma(a)$, as usual. The sign $\varnothing$ is used to symbolise empty sets ($p_\sigma = 0$ or $q_\sigma = 0$). In the designation of ${}^N F$, either the p, q or the $\boldsymbol{a}$, $\boldsymbol{c}$ symbols are occasionally omitted and the economical notation ${}^N F^{p_0, p_s}_{q_0, q_s}$ for the series (1) is sometimes used.

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Since many particular cases of the functions ${}^NF$ have been studied in detail, it would be useful to have the liberty to reduce a series ${}^NF$, by analogy with equation (3), to simpler functions ${}^NF$ with known special properties (intermediate reduction) rather than to functions ${}^1F \equiv F^p_q$ (complete reduction) as it takes place in equation (2).

To start with, we note that, accounting for the Leibnitz rule for the product differentiation, equation (2) can be transformed as

$$ {}^NF^{p_0,p_s}_{q_0,q_s} = F\begin{bmatrix} \boldsymbol{a}_0;\, \partial/\partial t_1 + \ldots \partial/\partial t_N \\ \boldsymbol{c}_0 \end{bmatrix} \prod_{s=1}^{N} F\begin{bmatrix} \boldsymbol{a}_s;\, x_s t_s \\ \boldsymbol{c}_s \end{bmatrix}\Bigg|_{\forall t_s=0}. \tag{4} $$

Both the GHS-1 of the sum of N variables, and the product of the GHS-1 in (4) can be written down as particular functions ${}^NF$ with omitted individual and gluing parameters, respectively (Niukkanen 1983)

$$ {}^NF^{p_0,p_s}_{q_0,q_s} = {}^NF\begin{bmatrix} \boldsymbol{a}_0\colon \varnothing;\ldots;\varnothing;\, \partial/\partial t_1,\ldots,\partial/\partial t_N \\ \boldsymbol{c}_0\colon \varnothing;\ldots;\varnothing \end{bmatrix} $$

$$ \times\, {}^NF\begin{bmatrix} \varnothing\colon \boldsymbol{a}_1;\ldots;\boldsymbol{a}_N;\, x_1t_1,\ldots,x_nt_N \\ \varnothing\colon \boldsymbol{c}_1;\ldots;\boldsymbol{c}_N \end{bmatrix}\Bigg|_{\forall t_s=0}. \tag{5} $$

It is, evidently, worth examination, whehter such type of relation will hold for the *unrestricted* ${}^NF$ functions in the right-hand side of (5).

To 'algebrise' our approach, introduce the Ω-product $\Phi * \Psi$ by

$$ \langle \Phi * \Psi | x_1,\ldots,x_N\rangle = \Phi(\partial/\partial t_1,\ldots,\partial/\partial t_N)\Psi(x_1t_1,\ldots,x_Nt_N)|_{\forall t_s=0}. \tag{6} $$

Then

$$ {}^NF^{p_0,p_s}_{q_0,q_s} = {}^NF^{p_0,0}_{q_0,0} * {}^NF^{0,p_s}_{0,q_s}. \tag{7} $$

Note two more relations

$$ {}^NF^{p_0'+p_0'',0}_{q_0'+q_0'',0} = {}^NF^{p_0',0}_{q_0',0} * {}^NF^{p_0'',0}_{q_0'',0} \tag{8} $$

and

$$ {}^NF^{0,p_s'+p_s''}_{0,q_s'+q_s''} = {}^NF^{0,p_s'}_{0,q_s'} * {}^NF^{0,p_s''}_{0,q_s''}, \tag{9} $$

which are obvious paraphrases of the particular operator factorisation formula (3). The relations (7), (8) and (9) cover all possible Ω-products of the special functions ${}^NF$ with omitted parameters.

To derive the generalised operator reduction formula we shall use, along with equations (7), (8) and (9), *associativity* and *commutativity* of the Ω-multiplication. The proof of these properties may be restricted, for brevity, to the case of one variable, without loss of generality. First, consider the relations

$$ \langle f*(\varphi*\psi)|x\rangle = f(\partial/\partial u)\langle \varphi*\psi|xu\rangle_{u=0} = f(\partial/\partial u)\varphi(\partial/\partial t)\psi(xtu)|_{t=u=0} \tag{10} $$

and

$$ \langle (f*\varphi)*\psi|x\rangle = \langle f*\varphi|\partial/\partial t\rangle\psi(xt)|_{t=0} = f(\partial/\partial u)\varphi(u\partial/\partial t)\psi(xt)|_{t=u=0}. \tag{11} $$

The evident identity

$$ \varphi(\partial/\partial t)\psi(xt) = \varphi(x\partial/\partial\tau)\psi(\tau)|_{\tau=xt} $$

implies the following simple yet very important *coupling rule*

$$ \varphi(\partial/\partial t)\psi(xt)|_{t=0} = \varphi(x\partial/\partial t)\psi(t)|_{t=0}, \tag{12} $$

which shows that, if the operators $\partial/\partial t$ and t involved in the arguments of two functions are coupled by the condition $t=0$, then the coefficients independent of t can be transferred from one argument to another.

Due to the coupling rule (12) the expressions (10) and (11) become equivalent which implies associativity of Ω-multiplication

$$f*(\varphi*\psi)=(f*\varphi)*\psi. \tag{13}$$

The evident identities

$$\begin{aligned}\langle\varphi*\psi|x\rangle &\equiv \varphi(\partial/\partial t)\psi(xt)|_{t=0}\\ &= \varphi(\partial/\partial t)\psi(\partial/\partial u)\exp(xtu)|_{t=u=0}\\ &= \psi(\partial/\partial u)\varphi(\partial/\partial t)\exp(xtu)|_{t=u=0}\\ &= \psi(\partial/\partial u)\varphi(xu)|_{u=0}\equiv\langle\psi*\varphi|x\rangle\end{aligned}$$

prove commutativity of Ω-multiplication

$$\varphi*\psi=\psi*\varphi. \tag{14}$$

With the help of (7), (8), (9), (13) and (14) we make the following transformations† (due to associativity of Ω-multiplication we can proceed without parentheses in Ω-products):

$$\begin{aligned} {}^N F^{p_0',p_s'}_{q_0',q_s'} * {}^N F^{p_0'',p_s''}_{q_0'',q_s''} &= {}^N F^{p_0',0}_{q_0',0} * {}^N F^{0,p_s'}_{0,q_s'} * {}^N F^{p_0'',0}_{q_0'',0} * {}^N F^{0,p_s''}_{0,q_s''} &(15)\\ &= {}^N F^{p_0',0}_{q_0',0} * {}^N F^{p_0'',0}_{q_0'',0} * {}^N F^{0,p_s'}_{0,q_s'} * {}^N F^{0,p_s''}_{0,q_s''} &(16)\\ &= {}^N F^{p_0'+p_0'',0}_{q_0'+q_0'',0} * {}^N F^{0,p_s'+p_s''}_{0,q_s'+q_s''} &(17)\\ &= {}^N F^{p_0'+p_0'',p_s'+p_s''}_{q_0'+q_0'',q_s'+q_s''}, &(18)\end{aligned}$$

which result in the *generalised operator reduction formula*

$${}^N F^{p_0'+p_0'',p_s'+p_s''}_{q_0'+q_0'',q_s'+q_s''} = {}^N F^{p_0',p_s'}_{q_0',q_s'} * {}^N F^{p_0'',p_s''}_{q_0'',q_s''}, \tag{19}$$

or, in detailed notation,

$$\begin{aligned} &{}^N F\left[\begin{matrix} \boldsymbol{a}_0', \boldsymbol{a}_0'': \boldsymbol{a}_1', \boldsymbol{a}_1''; \ldots; \boldsymbol{a}_N', \boldsymbol{a}_N''; x_1, \ldots, x_N \\ \boldsymbol{c}_0', \boldsymbol{c}_0'': \boldsymbol{c}_1', \boldsymbol{c}_1''; \ldots; \boldsymbol{c}_N', \boldsymbol{c}_N'' \end{matrix}\right] \\ &\qquad = {}^N F\left[\begin{matrix} \boldsymbol{a}_0': \boldsymbol{a}_1'; \ldots; \boldsymbol{a}_N'; \partial/\partial t_1, \ldots, \partial/\partial t_N \\ \boldsymbol{c}_0': \boldsymbol{c}_1'; \ldots; \boldsymbol{c}_N' \end{matrix}\right] \\ &\qquad\quad \times {}^N F\left[\begin{matrix} \boldsymbol{a}_0'': \boldsymbol{a}_1''; \ldots; \boldsymbol{a}_N''; x_1 t_1, \ldots, x_N t_N \\ \boldsymbol{c}_0'': \boldsymbol{c}_1''; \ldots; \boldsymbol{c}_N'' \end{matrix}\right]\Bigg|_{\forall t_s=0}. \end{aligned} \tag{20}$$

Equation (20) gives us a possibility of reducing a function ${}^N F$, by arbitrary breaking up each of the $2N+2$ sets of the gluon ($\boldsymbol{a}_0$, $\boldsymbol{c}_0$) and individual ($\boldsymbol{a}_s$, $\boldsymbol{c}_s$) parameters into two subsets, to two simpler functions ${}^N F'$ and ${}^N F''$. The initial operator reduction formula (2) becomes the evident particular case ($p_0''=q_0''=p_s'=q_s'=0$; $s=1,2,\ldots,N$)

† (7) → (15); (14) → (16); (8) and (9) → (17); (7) → (18).

of the generalised formula (20). Note two other particular cases of equation (20): firstly

$$F\begin{bmatrix} a_0'; u\partial/\partial t \\ c_0' \end{bmatrix} {}^N F\begin{bmatrix} a_0: a_1; \ldots; a_N; x_1 t, \ldots, x_N t \\ c_0: c_1; \ldots; c_N \end{bmatrix}$$

$$= {}^N F\begin{bmatrix} a_0, a_0': a_1; \ldots; a_N; ux_1, \ldots, ux_N \\ c_0, c_0': c_1; \ldots; c_N \end{bmatrix} \tag{21}$$

and secondly

$$F\begin{bmatrix} a_s'; u\partial/\partial t \\ c_s' \end{bmatrix} {}^N F\begin{bmatrix} a_0: a_1; \ldots; a_N; x_1, \ldots, x_s t, \ldots, x_N \\ c_0: c_1; \ldots; c_N \end{bmatrix}\Bigg|_{t=0}$$

$$= {}^N F\begin{bmatrix} a_0: a_1; \ldots; a_s, a_s'; \ldots; a_N; x_1, \ldots, x_1 u, \ldots, x_N \\ c_0; c_1; \ldots; c_s, c_s'; \ldots; c_N \end{bmatrix}. \tag{22}$$

Equations (21) and (22) allow us to change, selectively, either gluon parameters or individual ones and to introduce arbitrary scaling multipliers into the arguments of ${}^N F$ functions.

Formula (20) can be proved in a shorter way if use is made of the definition (1) for each of the ${}^N F$ functions on the right-hand side of equation (20). It should then be taken into account, that the relation

$$(\partial^{i_s'}/\partial t_s^{i_s'}) t_s^{i_s}|_{t_s=0} = i_s! \delta(i_s, i_s'),$$

where i_s and i_s' are summation indices and $\delta(i, i')$ is the Kroneker delta, holds for each of the differentiation variables t_s. However, the method exposed in § 4 has an advantage in that we do not have to recourse to explicit series representation for ${}^N F$ at any step†. Therefore, the approach not only helps us to ascertain useful general properties of the functions ${}^N F$, but also allows us to resort to an algebraic point of view, thus giving rise to a *group-theoretical interpretation* of the reduction formulae. Indeed, the operator reduction equation (19) implies, algebraically, that the set of ${}^N F$ elements is *closed* relative to Ω-multiplication. Since the Ω-multiplication is associative, the existence of unit and inverse elements remains to be checked in order to verify whether the set of the elements ${}^N F$ forms a group with respect to Ω-multiplication. Consider the function $e \equiv \exp(x_1 + x_2 + \ldots + x_N)$ which belongs, evidently, to ${}^N F$ set because

$$e \equiv \exp(x_1 + \ldots + x_N) = {}^N F_{0,0}^{0,0}\begin{bmatrix} \varnothing: \varnothing; x_s \\ \varnothing: \varnothing \end{bmatrix}.$$

The element e plays the part of the Ω-unit. Indeed, on one hand, e is the right unit ($\varphi * e = \varphi$) due to the evident identity $\varphi(x)\exp(xt) = \varphi(\partial/\partial t)\exp(xt)$ (we again restrict ourselves to the case of one variable). On the other hand, e is the left unit ($e * \varphi = \varphi$) because $\exp(\partial/\partial t)$ is a shift operator for any analytical function $\varphi(t)$. Therefore

$$\varphi * e = e * \varphi = \varphi. \tag{23}$$

† The possibility of avoiding explicit series representation of hypergeometric functions is the advantage inherent not only in the above reasoning, but it is one of the main technical advantages of the operator method on the whole, since the absence of the Pochhammer symbols conglomerations and elimination of tiresome calculations needed, as a rule, to transform these symbols at intermediate steps, give the method the transparent and apprehensible structure which is, usually, hard to achieve in other approaches.

The existence of the inverse element is implied by the formal identity

$$ {}^{N}F^{p_0,p_s}_{q_0,q_s} * {}^{N}F^{q_0,q_s}_{p_0,p_s} = {}^{N}F^{p_0+q_0,p_s+q_s}_{q_0+p_0,q_s+p_s} = {}^{N}F^{0,0}_{0,0} \equiv e. \tag{24} $$

Taking into account the relations (13), (14), (19), (23) and (24), one could say about the *commutative hypergeometric group* $\Omega_N({}^{N}F \in \Omega_N)$ if only such a standpoint would result in some non-trivial useful corollaries.

Returning to the primary interpretation of equation (20) as a convenient method of representing ${}^{N}F$ in a class of simpler functions of the same structure, one could regard such a view point as being remotely connected with some earlier attempts to relate certain types of multiple hypergeometric series to the products of GHS-1. For example, Appell and Kampé de Fériet (1926) made an effort to relate the Lauricella functions to the products of the Gauss function F^2_1, with the aid of qualitative argument connected with replacement of a part of individual parameters by gluon parameters. Though such a 'constructive' approach based on a speculative 'redistribution' of indices of Pochhammer symbols proved to be very useful for certain classification problems (Karlsson 1976), the absence of an appropriate technical background did not permit the 'genealogical' approach to become a general *analytical method* that would be suitable for a wide range of problems. The operator factorisation method may be considered as a far reaching development and generalisation of the earlier 'genealogical attempts' (Appell and Kampé de Fériet 1926, Karlsson 1976) in that it gives a rigorous analytical implementation of intuitional arguments connected with 'displacements' and 'redistributions' of indices in Pocchammer symbols. In particular, the simplest types of replacements relevant to descriptive constructions of Appell and Kampé de Fériet (1926) are implemented by the following analytical expressions

$$ {}^{N}F\begin{bmatrix} a\colon \varnothing; \ldots ; \varnothing \\ \varnothing\colon a_1; \ldots ; \varnothing \end{bmatrix} * \prod_{s=1}^{N} F^2_1\begin{bmatrix} a_s, b_s \\ c_s \end{bmatrix} \qquad \text{and} \qquad {}^{N}F\begin{bmatrix} \varnothing\colon c_1; \ldots ; c_N \\ c\colon \varnothing; \ldots ; \varnothing \end{bmatrix} * \prod_{s=1}^{N} F^2_1\begin{bmatrix} a_s, b_s \\ c_s \end{bmatrix}. $$

Somewhat more complicated expressions of similar structure can be obtained for more general hypergeometric series introduced by Karlsson (1976).

We conclude with the observation that operator factorisation method might prove rather useful in systematisation and classification of numerous relations obtained in special functions theory. Many of the seemingly independent theorems prove to be Ω-equivalent, that is connected by a certain Ω-multiplication procedure. To have an illustrative example, consider the evident operator relation between Appell and Humbert functions, $F_2 \equiv {}^{2}F^{1,1}_{0,1}$ and $\Psi_2 \equiv {}^{2}F^{1,0}_{0,1}$

$$ {}^{2}F^{1,1}_{0,1}\begin{bmatrix} \alpha\colon a_1; a_2 \\ \varnothing\colon c_1; c_2 \end{bmatrix} = {}^{2}F^{1,0}_{0,1}\begin{bmatrix} \alpha\colon \varnothing; \varnothing \\ \varnothing\colon c_1; c_2 \end{bmatrix} * \left(F^1_0\begin{bmatrix} a_1 \\ * \end{bmatrix} F^1_0\begin{bmatrix} a_2 \\ * \end{bmatrix} \right). \tag{25} $$

If $c_1 = c_2 = \alpha$, then (Burchnall and Chaundy 1941, equation (83)†)

$$ {}^{2}F^{1,0}_{0,1}\begin{bmatrix} \alpha\colon \varnothing;\ \varnothing; x_1, x_2 \\ \varnothing\colon \alpha; \alpha \end{bmatrix} = \exp(x_1 + x_2) F^0_1\begin{bmatrix} \varnothing; x_1 x_2 \\ \alpha \end{bmatrix}. \tag{26} $$

† Equation (26) is a corollary of more general relation

$$ {}^{2}F^{1,00}_{0,11}\begin{bmatrix} \alpha\colon \varnothing; \varnothing; x_1, x_2 \\ \varnothing\colon c_1; \alpha \end{bmatrix} = \exp(x_2)\, {}^{2}F^{1,00}_{1,10}\begin{bmatrix} \alpha\colon \varnothing; \varnothing; x_1 x_2, x_1 \\ c_1\colon \alpha; \varnothing \end{bmatrix} $$

which seems to be missing in earlier papers. In case $c_1 = \alpha$ the series ${}^{2}F$ in the right-hand side of the equation breaks up into the product of two GHS-1, thus giving rise to equation (26).

Substituting (26) into (25) we obtain

$$
\begin{aligned}
&{}^{2}F^{1,1}_{0,1}\left[\begin{matrix}\alpha:\, a_1;\, a_2;\, x_1, x_2\\ \varnothing:\, \alpha;\, \alpha\end{matrix}\right]\\
&\qquad = F^{0}_{1}\left[\begin{matrix}\varnothing;\, \partial/\partial t_1\, \partial/\partial t_2\\ \alpha\end{matrix}\right]\exp\left(\frac{\partial}{\partial t_1}+\frac{\partial}{\partial t_2}\right)F^{1}_{0}\left[\begin{matrix}a_1;\, x_1 t_1\\ \varnothing\end{matrix}\right]F^{1}_{0}\left[\begin{matrix}a_2;\, x_2 t_2\\ \varnothing\end{matrix}\right]\Bigg|_{t_1=t_2=0}\\
&\qquad = \sum_i \frac{1}{(\alpha)_i i!}\frac{\partial^i}{\partial t_1^i}\frac{\partial^i}{\partial t_2^i}F^{1}_{0}\left[\begin{matrix}a_1;\, x_1 t_1 + x_1\\ \varnothing\end{matrix}\right]F^{1}_{0}\left[\begin{matrix}a_2;\, x_2 t_2 + x_2\\ \varnothing\end{matrix}\right]\Bigg|_{t_1=t_2=0}\\
&\qquad = \sum_i \frac{1}{(\alpha)_i i!}x_1^i x_2^i (a_1)_i (a_2)_i F^{1}_{0}\left[\begin{matrix}a_i + i;\, x_1\\ \varnothing\end{matrix}\right]F^{1}_{0}\left[\begin{matrix}a_2 + i;\, x_2\\ \varnothing\end{matrix}\right],
\end{aligned}
$$

where from the known reduction formula (Erdélyi *et al* 1953, relation 5.10(3))

$$
{}^{2}F^{1,1}_{0,1}\left[\begin{matrix}\alpha:\, a_1, a_2;\, x_1, x_2\\ \varnothing:\, \alpha, \alpha\end{matrix}\right] = F^{1}_{0}\left[\begin{matrix}a_1;\, x_1\\ \varnothing\end{matrix}\right]F^{1}_{0}\left[\begin{matrix}a_2;\, x_2\\ \varnothing\end{matrix}\right]F^{2}_{1}\left[\begin{matrix}a_1, a_2;\, \dfrac{x_1 x_2}{(1-x_1)(1-x_2)}\\ \alpha\end{matrix}\right] \tag{27}
$$

follows. Therefore, the reduction formulae (26) and (27) prove to be Ω-equivalent.

The operator approach allows us not only to obtain new results along the lines of the general reasoning given in (Niukkanen 1983), but also gives a convenient background for better orientation in a mass of formulae obtained in special functions theory. The operator factorisation method might prove, apparently, especially useful for physical applications, since it allows us to use, instead of numerous conventional approaches which are, as a rule, too special or, on the contrary, too general, a single simple and universal approach which takes into account, in a comprehensive and transparent way, all specific features inherent in the formal structure of hypergeometric series.

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